The lattice Schwarzian KdV equation and its symmetries

2007 J. Phys. A: Math. Theor. 40 12753

(http://iopscience.iop.org/1751-8121/40/42/S18)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 12753–12761 doi:10.1088/1751-8113/40/42/S18

The lattice Schwarzian KdV equation and its symmetries

D Levi[1,2], M Petrera[1,3] and C Scimiterna[1,2,4]

[1] Dipartimento di Ingegneria Elettronica, Università degli Studi Roma Tre, Via della Vasca Navale 84, 00146 Roma, Italy
[2] Sezione INFN, Roma Tre, Via della Vasca Navale 84, 00146 Roma, Italy
[3] Zentrum Mathematik, Technische Universität München, Boltzmannstr. 3, D-85747 Garching bei München, Germany
[4] Dipartimento di Fisica E Amaldi, Università degli Studi Roma Tre, Via della Vasca Navale 84, 00146 Roma, Italy

E-mail: levi@fis.uniroma3.it, petrera@ma.tum.de and scimiterna@fis.uniroma3.it

Received 15 January 2007, in final form 4 April 2007
Published 2 October 2007
Online at stacks.iop.org/JPhysA/40/12753

Abstract

In this paper, we present a set of results on the symmetries of the lattice Schwarzian Korteweg–de Vries (lSKdV) equation. We construct the Lie point symmetries and, using its associated spectral problem, an infinite sequence of generalized symmetries and master symmetries. We finally show that we can use master symmetries of the lSKdV equation to construct non-autonomous non-integrable generalized symmetries.

PACS numbers: 02.30.Ik, 02.30.Ks

1. Introduction

The lattice version of the Schwarzian Korteweg–de Vries (lSKdV) equation

$$w_t = w_{xxx} - \frac{3w_{xx}^2}{2w_x} \tag{1}$$

is given by the nonlinear partial difference equation [18, 20]

$$\mathbb{Q} \doteq \alpha_1(x_{n,m} - x_{n,m+1})(x_{n+1,m} - x_{n+1,m+1}) - \alpha_2(x_{n,m} - x_{n+1,m})(x_{n,m+1} - x_{n+1,m+1}) = 0. \tag{2}$$

Equation (2) involves just four lattice points which lay on two orthogonal infinite lattices and are situated at the vertices of an elementary square. Here, $x_{n,m} : \mathbb{Z}^2 \to \mathbb{R}$ and α_1 and α_2 are two distinct real parameters.

Equation (2) is a lattice equation on *quad-graphs* belonging to the recent classification presented in [3], where the $3D$ *consistency* is used as a tool to establish its integrability. Precisely it is the equation Q_1, with $\delta = 0$, of the *Q-list* in [3].

As far as we know, the lSKdV equation was introduced for the first time by Nijhoff, Quispel and Capel [20] in 1983. A review of results about the lSKdV equation can be found in [16, 18].

Since we have two discrete independent variables, i.e. n and m, we can perform the continuous limit in two steps. Each step is achieved by shrinking the corresponding lattice step to zero and sending to infinity the number of points of the lattice.

In the first step, setting $\alpha_1 \doteq q^2, \alpha_2 \doteq p^2$, we define $x_{n,m} \doteq \widetilde{x}_k(\tau)$, where $k \doteq n + m$ and $\tau \doteq \delta m$, being $\delta \doteq p - q$. Considering the limits $m \to \infty, \delta \to 0$, we get the differential-difference equation

$$\frac{\mathrm{d}\widetilde{x}_k(\tau)}{\mathrm{d}\tau} = \frac{2(\widetilde{x}_{k+1} - \widetilde{x}_k)(\widetilde{x}_{k-1} - \widetilde{x}_k)}{p(\widetilde{x}_{k-1} - \widetilde{x}_{k+1})}. \tag{3}$$

The second continuous limit of equation (2) is performed by taking in equation (3) $\widetilde{x}_k(\tau) \doteq w(x, t)$, with $x \doteq 2(k + \tau/p)/p$ and $t \doteq 2(k/3 + \tau/p)/p^3$. If we carry out the limit $p \to \infty, k \to \infty$ and $\tau \to \infty$ in such a way that x and t remain finite, then equation (3) is transformed into the continuous Schwarzian KdV equation (1).

Integrable equations possess an infinite set of symmetries. Few of them are *point symmetries*, i.e. symmetries whose infinitesimal generators depend just on the independent and dependent variables, while an infinite number of them are *generalized symmetries*. These latter ones depend also on the derivatives of the dependent variable, for the continuous independent variables, and on a few lattice points, if the independent variables are discrete. The presence of this infinite Lie algebra of symmetries is one of the most important features of the integrability of a given nonlinear equation and it was used with profit in the past to provide integrability tests for several partial differential equations in $\mathbb{R}^2$ and differential-difference equations [14, 23].

To construct an integrability test based on symmetries, one needs to understand the structure of the infinite dimensional symmetry algebra of the integrable equations. In the case of completely discrete equations, the situation is not as clear as for the differential-difference or the partial-differential case. Results in this direction were obtained some time ago for the discrete-time Toda lattice [11] and recently for the lattice potential Korteweg–de Vries equation [12]. However, the Toda lattice and the lattice potential KdV equation are just examples and more examples are needed to get a sufficiently general idea of the possible stuctures which may appear.

The present paper is part of this research and is devoted to the study of the lSKdV equation exactly from this point of view. In section 2, we present some old and new results on the integrability of the lSKdV equation. Section 3 is devoted to the construction of its Lie point symmetries while in section 4 we consider its generalized symmetries. Finally, section 5 contains some concluding remarks.

2. The integrability of the lSKdV equation

Equation (2) has been obtained first by the direct linearization method [20]. In [18] one can find its associated spectral problem, which, as this equation is part of the Adler–Bobenko–Suris classification, can be obtained using a well-defined procedure [4, 5, 17].

Its Lax pair is given by the following overdetermined system of matrix equations for the vector $\Psi^{\lambda}_{n,m} \doteq \left(\psi^1_{n,m}(\lambda), \psi^2_{n,m}(\lambda)\right)^{\mathrm{T}}, \lambda \in \mathbb{C}$,

$$\Psi^{\lambda}_{n+1,m} = L^{\lambda}_{n,m}\Psi^{\lambda}_{n,m}, \tag{4a}$$

$$\Psi^{\lambda}_{n,m+1} = M^{\lambda}_{n,m}\Psi^{\lambda}_{n,m}, \tag{4b}$$

where

$$L^{\lambda}_{n,m} \doteq \begin{pmatrix} 1 & x_{n,m} - x_{n+1,m} \\ \lambda\alpha_1(x_{n,m} - x_{n+1,m})^{-1} & 1 \end{pmatrix}$$

and

$$M^{\lambda}_{n,m} \doteq \begin{pmatrix} 1 & x_{n,m} - x_{n,m+1} \\ \lambda\alpha_2(x_{n,m} - x_{n,m+1})^{-1} & 1 \end{pmatrix}.$$

The consistency of equations (4) implies the discrete Lax equation

$$L^{\lambda}_{n,m+1} M^{\lambda}_{n,m} = M^{\lambda}_{n+1,m} L^{\lambda}_{n,m}.$$

We can rewrite equations (4) in scalar form in terms of $\psi_{n,m} \doteq \psi^2_{n,m}(\lambda)$:

$$(x_{n+2,m} - x_{n+1,m})\psi_{n+2,m} - (x_{n+2,m} - x_{n,m})\psi_{n+1,m} + (1 - \lambda\alpha_1)(x_{n+1,m} - x_{n,m})\psi_{n,m} = 0, \quad (5a)$$

$$(x_{n,m+2} - x_{n,m+1})\psi_{n,m+2} - (x_{n,m+2} - x_{n,m})\psi_{n,m+1} + (1 - \lambda\alpha_2)(x_{n,m+1} - x_{n,m})\psi_{n,m} = 0. \quad (5b)$$

It is worthwhile to observe here that equation (2) and the Lax equations (5) are invariant under the discrete symmetry obtained by interchanging n with m and α_1 with α_2.

To get meaningful Lax equations, the real field $x_{n,m}$ cannot go asymptotically to a constant but must be written as $x_{n,m} \doteq u_{n,m} + \beta_0 m + \alpha_0 n$, where α_0 and β_0 are constants related to α_1 and α_2 by the condition $\alpha_1\beta_0^2 = \alpha_2\alpha_0^2$. Under this transformation of the dependent variable, the lSKdV equation and its Lax pair can be respectively rewritten as

$$\begin{aligned} &\alpha_1(u_{n,m} - u_{n,m+1} - \beta_0)(u_{n+1,m} - u_{n+1,m+1} - \beta_0) \\ &\qquad = \alpha_2(u_{n,m} - u_{n+1,m} - \alpha_0)(u_{n,m+1} - u_{n+1,m+1} - \alpha_0) \end{aligned} \quad (6)$$

and

$$\left(1 + v^{(n)}_{n,m}\right)\psi_{n+2,m} - \left(2 + v^{(n)}_{n,m}\right)\psi_{n+1,m} + (1 - \lambda\alpha_1)\psi_{n,m} = 0, \quad (7a)$$

$$\left(1 + v^{(m)}_{n,m}\right)\psi_{n,m+2} - \left(2 + v^{(m)}_{n,m}\right)\psi_{n,m+1} + (1 - \lambda\alpha_2)\psi_{n,m} = 0, \quad (7b)$$

where

$$v^{(n)}_{n,m} \doteq \frac{u_{n+2,m} - 2u_{n+1,m} + u_{n,m}}{u_{n+1,m} - u_{n,m} + \alpha_0}, \qquad v^{(m)}_{n,m} \doteq \frac{u_{n,m+2} - 2u_{n,m+1} + u_{n,m}}{u_{n,m+1} - u_{n,m} + \beta_0},$$

where $u_{n,m} \to c$, with $c \in \mathbb{R}$, as n and m go to infinity.

One can construct the class of differential-difference equations associated with equation (7a) by requiring the existence of a set of operators M_n such that [7]

$$L_n\psi_n = \mu\psi_n, \qquad \frac{\mathrm{d}\psi_n}{\mathrm{d}t} = -M_n\psi_n, \qquad \frac{\mathrm{d}L_n}{\mathrm{d}t} = [L_n, M_n]$$

with $L_n \doteq \left(1 + v^{(n)}_{n,m}(t)\right)E^2 - \left(2 + v^{(n)}_{n,m}(t)\right)E$. Here E is the (positive) shift operator in the variable n. If $\mathrm{d}\mu/\mathrm{d}t = 0$ the class of differential-difference equations one so obtains will be called *isospectral*, while if $\mathrm{d}\mu/\mathrm{d}t \neq 0$ it will be called *non-isospectral*.

As the field $v^{(n)}_{n,m}(t)$ appears multiplying both E and E^2, the expression of the recursive operator turns out to be extremely complicated, containing triple sums and products of the dependent fields. So, to get a more manageable problem, we look for transformations of the spectral problem (7a) which reduce it to a simpler form in which the potential will appear just once. In the literature, one can find two different discrete spectral problems involving three lattice points which have been already solved. They are the discrete Schrödinger spectral problem introduced by Case [8],

$$\phi_{n-1} + a_n\phi_{n+1} + b_n\phi_n = \lambda\phi_n, \quad (8)$$

which is associated with the Toda and Volterra differential-difference equations [10] and the asymmetric discrete Schrödinger spectral problem introduced by Shabat [22] and Boiti *et al* [6],

$$\phi_{n+2} = \frac{2p}{s_n}\phi_{n+1} + \lambda\phi_n. \tag{9}$$

The latter one has been used to solve the so-called discrete KdV equation [15]. In equations (8) and (9), the functions a_n, b_n, s_n may depend parametrically on a *continuous variable t* but also on a *discrete variable m*. As all three spectral problems (7*a*), (8) and (9) involve just three points on the lattice, we can relate them by a gauge transformation $\psi_n \doteq f_n(\mu)g_n(\{u_{n,m}\})\phi_n$, where $\{u_{n,m}\} \doteq (u_{n,m}, u_{n\pm1,m}, u_{n,m\pm1}, \ldots)$. These mappings give rise to a Miura transformation between the involved fields. For instance, when we transform the spectral problem (7*a*) into the discrete Schrödinger spectral problem (8), we get

$$b_n \doteq b_{n,m} = 0, \tag{10a}$$

$$a_n \doteq a_{n,m} = \frac{4(u_{n+1,m} - u_{n,m} + \alpha_0)^2}{(u_{n+2,m} - u_{n,m} + 2\alpha_0)(u_{n+1,m} - u_{n-1,m} + 2\alpha_0)}. \tag{10b}$$

If we transform the spectral problem given in equation (7*a*) into the asymmetric discrete Schrödinger spectral problem (9), the relationship between the fields $s_n \doteq s_{n,m}$ and $u_{n,m}$ is more involved as it is expressed in terms of infinite products.

We will use in section 4 the transformation (10*b*) and the equivalent one between (7*b*) and (8) which will define a field $\widetilde{a}_m$ given by equation (10*b*) with n and m and α_0 and β_0 interchanged, to build the generalized symmetries of equation (6) from the nonlinear differential-difference equations associated with the spectral problem (8) [10] with $b_n = 0$.

3. Point symmetries of the lSKdV equation

Here we look for Lie point symmetries of equation (2), with $\alpha_1 \neq \alpha_2$, using the technique introduced in [13]. The symmetries we obtain in this way turn out to be the same as those for (6).

The Lie symmetries of the lSKdV equation (2) are given by those continuous transformations which leave the equation invariant. From the infinitesimal point of view, they are obtained by requiring the infinitesimal invariant condition

$$\mathrm{pr}\,\widehat{X}_{n,m}\mathbb{Q}|_{\mathbb{Q}=0} = 0, \tag{11}$$

where, as we keep the form of the lattice invariant,

$$\widehat{X}_{n,m} = \Phi_{n,m}(x_{n,m})\partial_{x_{n,m}}. \tag{12}$$

By $\mathrm{pr}\,\widehat{X}_{n,m}$ we mean the prolongation of the infinitesimal generator $\widehat{X}_{n,m}$ to the other three points appearing in $\mathbb{Q} = 0$, i.e. $x_{n+1,m}$, $x_{n,m+1}$ and $x_{n+1,m+1}$.

By solving the equation $\mathbb{Q} = 0$ w.r.t. $x_{n+1,m+1}$ and substituting it in equation (11) we get a functional equation for $\Phi_{n,m}(x_{n,m})$. Looking at its solutions in the form $\Phi_{n,m}(x_{n,m}) = \sum_{k=0}^{\gamma}\Phi_{n,m}^{(k)}x_{n,m}^k$, $\gamma \in \mathbb{N}$, we see that in order to balance the leading order in $x_{n,m}$, γ cannot be greater than 2. Equating now to zero the coefficients of the powers of $x_{n,m}$, $x_{n+1,m}$ and $x_{n,m+1}$, we get an overdetermined system of determining equations. Solving the resulting difference equations, we find that the functions $\Phi_{n,m}^{(i)}$'s, $i = 0, 1, 2$, must be constants. Hence the infinitesimal generators of the algebra of Lie point symmetries are given by

$$\widehat{X}_{n,m}^{(0)} = \partial_{x_{n,m}}, \qquad \widehat{X}_{n,m}^{(1)} = x_{n,m}\partial_{x_{n,m}}, \qquad \widehat{X}_{n,m}^{(2)} = x_{n,m}^2\partial_{x_{n,m}}.$$

The generators $\widehat{X}^{(i)}_{n,m}$, $i = 0, 1, 2$, span the Lie algebra $\mathfrak{sl}(2)$:

$$\left[\widehat{X}^{(0)}_{n,m}, \widehat{X}^{(1)}_{n,m}\right] = \widehat{X}^{(0)}_{n,m}, \qquad \left[\widehat{X}^{(1)}_{n,m}, \widehat{X}^{(2)}_{n,m}\right] = \widehat{X}^{(2)}_{n,m}, \qquad \left[\widehat{X}^{(0)}_{n,m}, \widehat{X}^{(2)}_{n,m}\right] = 2\widehat{X}^{(1)}_{n,m}.$$

We can write down the group transformation by integrating the differential equation

$$\frac{\mathrm{d}\widetilde{x}_{n,m}(\epsilon)}{\mathrm{d}\epsilon} = \Phi_{n,m}(\widetilde{x}_{n,m}(\epsilon)), \tag{13}$$

with the initial condition $\widetilde{x}_{n,m}(\epsilon = 0) = x_{n,m}$. In this way we get the Möbius transformation [4, 5, 18]

$$\widetilde{x}_{n,m}(\epsilon_0, \epsilon_1, \epsilon_2) = \frac{(\epsilon_0 + x_{n,m})\,\mathrm{e}^{\epsilon_1}}{1 - \epsilon_2(\epsilon_0 + x_{n,m})\,\mathrm{e}^{\epsilon_1}},$$

where the ϵ_i's are the parameters associated with the infinitesimal generators $\widehat{X}^{(i)}$, $i = 0, 1, 2$.

We finally notice that, in the case when $\alpha_1 = \alpha_2$, equation (2) simplifies to the product of two linear discrete wave equations:

$$(x_{n,m} - x_{n+1,m+1})(x_{n+1,m} - x_{n,m+1}) = 0,$$

which is trivially solved by taking $x_{n,m} = f_{n\pm m}$ and the Lie point symmetries belong to an infinite dimensional Lie algebra.

4. Generalized symmetries of the lSKdV equation

A generalized symmetry is obtained when the function $\Phi_{n,m}$ appearing in equation (12) depends on $\{x_{n,m}\}$ and not only on $x_{n,m}$. A way to obtain it is to look at those differential-difference equations (13) associated with the spectral problem (8) which are compatible with equation (2). From equations (10), we see that the lSKdV equation can be associated with the discrete Schrödinger spectral problem when $b_{n,m} = 0$, i.e. when the associated hierarchy of differential-difference equations is given by the Volterra hierarchy [10]. So, applying the Miura transformation (10) to the differential-difference equations of the Volterra hierarchy, we can obtain the symmetries of the lSKdV equation. The Miura transformation (10) preserves the integrability of the Volterra hierarchy if $u_{n,m} \to c$, with $c \in \mathbb{R}$, as n and m go to infinity.

The procedure to get the generalized symmetries for the lSKdV equation is better shown in a specific example, the case of the Volterra equation, an isospectral deformation of the spectral problem (8):

$$\frac{\mathrm{d}a_{n,m}(\epsilon_0)}{\mathrm{d}\epsilon_0} = a_{n,m}(a_{n+1,m} - a_{n-1,m}). \tag{14}$$

Let us substitute the Miura transformation, given by equation (10*b*), into equation (14) and let us assume that

$$\frac{\mathrm{d}u_{n,m}(\epsilon_0)}{\mathrm{d}\epsilon_0} = F_{n,m}(u_{n-1,m}, u_{n,m}, u_{n+1,m}).$$

Equation (14) is thus a functional equation for $F_{n,m}$ which can be solved as we did in the previous section by comparing powers at infinity or by transforming it into an overdetermined system of linear partial differential equations [1, 2]. In this way we get, up to a point transformation,

$$F_{n,m} = \frac{4(u_{n,m} - u_{n-1,m} + \alpha_0)(u_{n,m} - u_{n+1,m} - \alpha_0)}{u_{n+1,m} - u_{n-1,m} + 2\alpha_0}. \tag{15}$$

Equation (15) is nothing else but equation (3). One can verify that equation (15) is a generalized symmetry of the lSKdV equation (6) by proving that $\Phi_{n,m} = F_{n,m}(u_{n-1,m}, u_{n,m}, u_{n+1,m})$ satisfies equation (11).

If we start from a higher equation of the isospectral Volterra hierarchy

$$\frac{\mathrm{d}a_{n,m}(\epsilon_1)}{\mathrm{d}\epsilon_1} = a_{n,m}[a_{n-1,m}(a_{n-2,m} + a_{n-1,m} + a_{n,m} - 4) - a_{n+1,m}(a_{n+2,m} + a_{n+1,m} + a_{n,m} - 4)],$$

we get a second generalized symmetry of the lSKdV equation requiring $F_{n,m} = F_{n,m}(u_{n-2,m}, u_{n-1,m}, u_{n,m}, u_{n+1,m}, u_{n+2,m})$. It reads

$$\begin{aligned} F_{n,m} = {} & \frac{(u_{n,m} - u_{n-1,m} + \alpha_0)(u_{n,m} - u_{n+1,m} - \alpha_0)}{(u_{n+1,m} - u_{n-1,m} + 2\alpha_0)^2} \\ & \times \left[\frac{(u_{n+2,m} - u_{n+1,m} + \alpha_0)(u_{n-1,m} - u_{n,m} - \alpha_0)}{u_{n+2,m} - u_{n,m} + 2\alpha_0} \right. \\ & \left. - \frac{(u_{n-1,m} - u_{n-2,m} + \alpha_0)(u_{n,m} - u_{n+1,m} - \alpha_0)}{u_{n,m} - u_{n-2,m} + 2\alpha_0} \right]. \end{aligned} \tag{16}$$

This procedure can be clearly carried out for any equation of the Volterra hierarchy [10] giving a hierarchy of symmetries for the lSKdV equation.

If we consider the non-isospectral hierarchy, the only local equation is [10]

$$\frac{\mathrm{d}a_{n,m}(\sigma)}{\mathrm{d}\sigma} = a_{n,m}[a_{n,m} - (n-1)a_{n-1,m} + (n+2)a_{n+1,m} - 4]$$

and it provides, up to a Lie point symmetry, two local equations

$$\frac{\mathrm{d}u_{n,m}(\sigma_0)}{\mathrm{d}\sigma_0} = u_{n,m} + \alpha_0 n, \tag{17}$$

$$\frac{\mathrm{d}u_{n,m}(\sigma_1)}{\mathrm{d}\sigma_1} = \frac{4n(u_{n,m} - u_{n-1,m} + \alpha_0)(u_{n,m} - u_{n+1,m} - \alpha_0)}{u_{n+1,m} - u_{n-1,m} + 2\alpha_0}. \tag{18}$$

Equation (17) is obtained as the multiplicative factor of an integration constant. One can easily show that equation (17) is not a symmetry of the lSKdV equation but it commutes with all its known symmetries. Equation (18) is a master symmetry [9]: it does not commute with the lSKdV equation but commuting it with equation (15) one gets equation (16) and commuting it with equation (16) one gets a higher order symmetry. So through it one can construct a hierarchy of generalized symmetries of the lSKdV equation.

In the construction of generalized symmetries for the differential-difference Volterra equation [10] one was able to construct a symmetry from the master symmetry (18) by combining it with the second isospectral symmetry (16) multiplied by t. This seems not to be the case for difference-difference equations. As was shown in [12], for the case of the lattice potential KdV equation, there is no combination of equation (18) with isospectral symmetries which gives a symmetry of equation (6).

As the lSKdV equation admits a discrete symmetry corresponding to an exchange of n with m and α_1 with α_2, one can construct another class of generalized and master symmetries by considering the equations obtained from the spectral problem (8) in the m lattice variable depending on the potential $\widetilde{a}_m$. In this way, we get

$$\frac{\mathrm{d}u_{n,m}(\widetilde{\epsilon}_0)}{\mathrm{d}\widetilde{\epsilon}_0} = \frac{4(u_{n,m} - u_{n,m-1} + \beta_0)(u_{n,m} - u_{n,m+1} - \beta_0)}{u_{n,m+1} - u_{n,m-1} + 2\beta_0},$$

$$\begin{aligned} \frac{\mathrm{d}u_{n,m}(\widetilde{\epsilon}_1)}{\mathrm{d}\widetilde{\epsilon}_1} = {} & \frac{(u_{n,m} - u_{n,m-1} + \beta_0)(u_{n,m} - u_{n,m+1} - \beta_0)}{(u_{n,m+1} - u_{n,m-1} + 2\beta_0)^2} \\ & \times \left[\frac{(u_{n,m+2} - u_{n,m+1} + \beta_0)(u_{n,m-1} - u_{n,m} - \beta_0)}{u_{n,m+2} - u_{n,m} + 2\beta_0} \right. \\ & \left. - \frac{(u_{n,m-1} - u_{n,m-2} + \beta_0)(u_{n,m} - u_{n,m+1} - \alpha_0)}{u_{n,m} - u_{n,m-2} + 2\beta_0} \right] \end{aligned}$$

and

$$\frac{\mathrm{d}u_{n,m}(\widetilde{\sigma}_0)}{\mathrm{d}\widetilde{\sigma}_0} = u_{n,m} + \beta_0 m,$$

$$\frac{\mathrm{d}u_{n,m}(\widetilde{\sigma}_1)}{\mathrm{d}\widetilde{\sigma}_1} = \frac{4m(u_{n,m} - u_{n,m-1} + \beta_0)(u_{n,m} - u_{n,m+1} - \beta_0)}{u_{n,m+1} - u_{n,m-1} + 2\beta_0}.$$

A different class of symmetries can be obtained applying the following theorem, introduced in [12], which provides a constructive tool to obtain generalized symmetries for the lSKdV equation (2).

Theorem 1. *Let* $\mathbb{Q}(u_{n,m}, u_{n\pm1,m}, u_{n,m\pm1}, \ldots; \alpha_1, \alpha_2) = 0$ *be an integrable partial difference equation invariant under the discrete symmetry* $n \leftrightarrow m, \alpha_1 \leftrightarrow \alpha_2$. *Let* $\widehat{Z}_n$ *be the differential operator*

$$\widehat{Z}_n \doteq Z_n(u_{n,m}, u_{n\pm1,m}, u_{n,m\pm1}, \ldots; \alpha_1, \alpha_2)\partial_{u_{n,m}},$$

such that

$$\mathrm{pr}\,\widehat{Z}_n\mathbb{Q}|_{\mathbb{Q}=0} = a g_{n,m}(u_{n,m}, u_{n\pm1,m}, u_{n,m\pm1}, \ldots; \alpha_1, \alpha_2),$$

where $g_{n,m}(u_{n,m}, u_{n\pm1,m}, u_{n,m\pm1}, \ldots; \alpha_1, \alpha_2)$ *is invariant under the discrete symmetry* $n \leftrightarrow m, \alpha_1 \leftrightarrow \alpha_2$ *and* a *is an arbitrary constant. Then*

$$\left(\frac{1}{a}\mathrm{pr}\,\widehat{Z}_n - \frac{1}{b}\mathrm{pr}\,\widehat{Z}_m\right)\mathbb{Q}\bigg|_{\mathbb{Q}=0} = 0,$$

where the operator $\widehat{Z}_m \doteq Z_m(u_{n,m}, u_{n,m\pm1}, u_{n\pm1,m}, \ldots; \alpha_2, \alpha_1)\partial_{u_{n,m}}$ *is obtained from* $\widehat{Z}_n$ *under* $n \leftrightarrow m, \alpha_1 \leftrightarrow \alpha_2$, *so that*

$$\mathrm{pr}\,\widehat{Z}_m\mathbb{Q}|_{\mathbb{Q}=0} = b g_{n,m}(u_{n,m}, u_{n\pm1,m}, u_{n,m\pm1}, \ldots; \alpha_2, \alpha_1),$$

b being a constant. So

$$\widehat{Z}_{n,m} \doteq \frac{1}{a}\widehat{Z}_n - \frac{1}{b}\widehat{Z}_m$$

is a symmetry of $\mathbb{Q} = 0$.

Using this theorem it is easy to show that from the master symmetry (18) we can construct a generalized symmetry, given by

$$\frac{\mathrm{d}u_{n,m}(\epsilon)}{\mathrm{d}\epsilon} = \frac{4n(u_{n,m} - u_{n-1,m} + \alpha_0)(u_{n,m} - u_{n+1,m} - \alpha_0)}{u_{n+1,m} - u_{n-1,m} + 2\alpha_0} + \frac{4m(u_{n,m} - u_{n,m-1} + \beta_0)(u_{n,m} - u_{n,m+1} - \beta_0)}{u_{n,m+1} - u_{n,m-1} + 2\beta_0}.$$

The above symmetry has been implicitly used, together with point symmetries, by Nijhoff and Papageorgiou [19] to perform the similarity reduction of the lSKdV equation and get a discrete analogue of the Painlevé II equation.

5. Conclusions

In this paper, we have constructed by group theoretical methods the symmetries of the lSKdV equation. The Lie point symmetry algebra provides a Möbius transformation of the dependent variable. The generalized symmetries are obtained in a constructive way by considering the spectral problem associated with the lSKdV equation. As was shown in [16, 18] we can associate with this integrable lattice equation an asymmetric discrete Schrödinger spectral problem with two potentials. Since the corresponding nonlinear evolution equations are too complicated we have transformed the obtained spectral problem into the standard discrete Schrödinger spectral problem, whose corresponding evolution equations are the Toda and Volterra hierarchies. In this way, we have constructed two classes of integrable symmetries, associated with the n and m components of the Lax pair and we have obtained symmetries starting from the master symmetries. So we have obtained in a group theoretical framework the nonlinear reductions considered in [19].

We left to future work to prove whether the two classes of isospectral symmetries are independent and whether the symmetries obtained from the master symmetries are integrable or not. Another open problem is the construction of the recursive operator for the lSKdV symmetries.

After we finished this paper, a referee pointed out to us the paper by Rasin and Hydon [21] submitted to *Stud. Appl. Math.* and available in the personal web page of the authors. In this paper, the authors construct the symmetries for all quad-graph equations contained in the Adler–Bobenko–Suris list [3] by solving the symmetry determining equations. In our work, we recover the same results presented in [21] for the Q_1 equation by considering isospectral and non-isospectral deformations of its Lax pair. In this way, we are able to answer some of the questions in their conclusions.

Acknowledgments

DL and MP were partially supported by PRIN Project 'SINTESI-2004' of the Italian Minister for Education and Scientific Research. This research is part of a joint Italian Russian research project 'Classification of integrable discrete and continuous models' financed by Consortium EINSTEIN and Russian Foundation for Basic Research. MP was partially supported by the European Community through the FP6 Marie Curie RTN ENIGMA (contract number MRTN-CT-2004-5652).

References

[1] Aczel J 1966 *Lectures on Functional Equations and Their Applications* (New York: Academic)
[2] Aczel J (ed) 1984 *Functional Equations: History, Applications and Theory* (Amsterdam: Reidel)
[3] Adler V E, Bobenko A I and Suris Yu B 2003 Classification of integrable equations on quad-graphs. The consistency approach *Commun. Math. Phys.* **233** 513–43
[4] Bobenko A I and Suris Yu B 2002 Integrable systems on quad-graphs *Int. Math. Res. Not.* **11** 573–611
[5] Bobenko A I and Suris Yu B 2002 Integrable noncommutative equations on quad-graphs. The consistency approach *Lett. Math. Phys.* **61** 241–54
[6] Boiti M, Pempinelli F, Prinari B and Spire A 2001 An integrable discretization of KdV at large times *Inverse Probl.* **17** 515–26
[7] Bruschi M and Ragnisco O 1980 Extension of the Lax method to solve a class of nonlinear evolution equations with x-dependent coefficients associated to the matrix Schrödinger spectral problem *Lett. Nuovo Cimento* **29** 327–30
[8] Case K M 1974 The discrete inverse scattering problem in one dimension *J. Math. Phys.* **15** 143–6
[9] Fuchssteiner B and Ma W-X 1991 An approach to master symmetries of lattice equations *Symmetries and Integrability of Difference Equations (Canterbury, 1996)* (Cambridge: Cambridge University Press) pp 247–60
[10] Heredero R H, Levi D, Rodriguez M A and Winternitz P 2000 Lie algebra contractions and symmetries of the Toda hierarchy *J. Phys. A: Math. Gen.* **33** 5025–40
[11] Levi D and Martina L 2001 Integrable hierarchies of nonlinear difference-difference equations and symmetries *J. Phys. A: Math. Gen.* **34** 10357–68
[12] Levi D and Petrera M 2007 Continuous symmetries of the lattice potential KdV equation *J. Phys. A: Math. Gen.* **40** 4141–59

[13] Levi D and Winternitz P 2006 Continuous symmetries of difference equations *J. Phys. A: Math. Gen.* **39** R1–63

[14] Mikhailov A V, Shabat A B and Yamilov R I 1987 A symmetric approach to the classification of nonlinear equations. Complete lists of integrable systems *Usp. Mat. Nauk* **42** 3–53

[15] Narita K 1998 Solutions for the Mikhailov–Shabat–Yamilov difference-differential equations and generalized solutions for the Volterra and the Toda lattice equations *Prog. Theor. Phys.* **99** 337–48

[16] Nijhoff F W 1997 On some "Schwarzian" equations and their discrete analogues. Algebraic aspects of integrable systems *Progr. Nonlinear Differential Equations Appl.* vol 26 (Boston, MA: Birkhauser Boston) pp 237–60

[17] Nijhoff F W 2002 Lax pair for the Adler (lattice Krichever-Novikov) system *Phys. Lett.* A **297** 49–58

[18] Nijhoff F W and Capel H W 1995 The discrete Korteweg-de Vries equation *Acta Appl. Math.* **39** 133–58

[19] Nijhoff F W and Papageorgiou V G 1991 Similarity reductions on integrable lattices and discrete analogues of the Painlevé II equation *Phys. Lett.* A **153** 337–44

[20] Nijhoff F W, Quispel G R W and Capel H W 1983 Direct linearization of nonlinear difference-difference equations *Phys. Lett.* A **97** 125–8

[21] Rasin O G and Hydon P E Symmetries of integrable difference equations on the quad-graph *Stud. Appl. Math.* submitted

[22] Shabat A B 2000 Dressing chains and lattices *Nonlinearity, Integrability and All That. Twenty Years After NEEDS 79* ed M Boiti, L Martina, F Pempinelli, B Prinari and G Soliani (River Edge, NJ: World Scientific) pp 331–42

[23] Yamilov R 2006 Symmetries as integrability criteria for differential-difference equations *J. Phys. A: Math. Gen.* **39** R541–623